

An exact string representation of 3d SU(2) lattice Yang–Mills theory

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We show that 3d SU(2) lattice Yang–Mills theory can be cast in the form of an exact string representation. The derivation starts from the exact dual (or spin foam) representation of the lattice gauge theory. We prove that every dual configuration (or spin foam) can be equivalently described as a self-avoiding worldsheet of strings on a framing of the lattice. Using this correspondence, we translate the partition function into a sum over closed worldsheets that are weighted with explicit amplitudes. The expectation value of two Polaykov loops with spin j becomes a sum over worldsheets that are bounded by $2j$ strings along a framing of the loops.

I. INTRODUCTION

It has been a long-standing conjecture that gauge theory has a dual or effective description in terms of string-like degrees of freedom (see e.g. [1, 2, 3, 4] for a review). The idea made its first appearance in the 60's when dual resonance models of hadron scattering were interpreted in terms of strings [5, 6, 7, 8, 9]. It reappeared again, when Wilson introduced the strong-coupling expansion and argued that confinement is an effect of flux lines between quarks [10]. This motivated attempts to find an exact or effective string representation of Yang–Mills theory, and led to the study of loop equations [11, 12], and to lattice and continuum models of the Nambu–Goto string (see e.g. [13, 14, 15] and refs. in [16]). Then, critical string theory was introduced [17, 18, 19], and developed further into superstring theory, with the broader aim of unifying gauge theory, matter and gravity. Another aspect of the gauge–string duality was revealed when 't Hooft analyzed the large N limit of perturbative Yang–Mills theory [20]. In the context of string theory, the idea was revived more recently by the AdS–CFT correspondence: by the conjecture that a supersymmetric conformal Yang–Mills theory has an equivalent description in terms of superstrings in an AdS spacetime [21, 22, 23, 24].

Conceptually, the present paper is close to Wilson's original approach, where flux lines arise as diagrams of a strong-coupling expansion. There are different versions of the strong-coupling expansion that have different convergence properties. Here, we are concerned with the “resummed” expansion that is convergent for any coupling [25, 26]: it results from an expansion of plaquette actions into a basis of characters, and from a subsequent integration over the connection. Thus, the sum over graphs is not an expansion in powers of β , but rather a *dual representation* that is equivalent to the original lattice gauge theory [27, 28, 29, 30]. For this reason, we try to avoid the adjective “strong-coupling” and call the graphs instead *spin foams* [30]. Originally, this name was introduced for SU(2) [31], but it is also used for general gauge groups. In the case of SU(2), one obtains a sum over spin assignments to the lattice that satisfy certain spin coupling conditions. Each admissible configuration is a spin foam.

To some extent, the concept of spin foams already embodies the idea of an exact gauge–string duality: spin foams can be considered as branched surfaces that are worldsheets of flux lines (see sec. 6.3 in [32] and [33]). Due to the branching and the labelling with representations, these surfaces are not worldsheets as in string theory, however.

The new element of this paper is the following: we show that in 3 dimensions spin foams of SU(2) can be decomposed into worldsheets that do not branch and carry no representation label. They can be regarded as worldsheets of strings in the fundamental representation. To carry out this decomposition, we have to apply two modifications to the lattice: the cubic lattice is replaced by a tessellation by cubes and truncated rhombic dodecahedra. This ensures that at every edge exactly three faces intersect. In the second step, the 2-skeleton of this lattice is framed (or thickened). The thickening allows us to replace each spin assignment j_f to a face by $2j_f$ sheets of a surface. We show that these sheets can be connected to form a worldsheet in the thickened complex. Moreover, by imposing suitable restrictions on the worldsheets, we can establish a bijection between spin foams and worldsheets.

Once this bijection is given, it is simple to rewrite exact sums over spin foams as exact sums over worldsheets. The boundary conditions depend on the observable that is computed by the spin foam sum. In the case of a Wilson loop

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in the representation j , the sum extends over worldsheets that are bounded by $2j$ closed strings. In this paper, we derive the sum over worldsheets explicitly for two Polyakov loops of spin j that run parallel through the lattice.

The paper is organized as follows: in section II we set our conventions for spin foams and their boundaries (so-called spin networks). Then, we specify 3d SU(2) lattice Yang–Mills theory with the heat kernel action (sec. III). In section IV, we describe the dual transform of the partition function and of the expectation value of two Polyakov loops. The central part of the paper is section V, where we introduce worldsheets on the framed lattice, and prove the bijection between worldsheets and spin foams. In the final section, we formulate both the partition function and the expectation value of the Polyakov loops as exact sums over worldsheets with explicit amplitude factors.

II. SPIN FOAMS AND SPIN NETWORKS

In this section, we set our conventions for spin foams and spin networks of SU(2). Spin networks formalize the concept of flux line, and spin foams can be regarded as worldsheets of these flux lines.

In this paper, spin foams will live on 3-complexes where at each interior edge exactly three faces meet. Spin networks will only lie on the boundary of this complex. For this reason, we do not need to consider the most general concept of spin foam and spin network that could occur and restrict ourselves to the following definition.

Let Λ be a complex where at each interior edge exactly three faces meet. A spin foam F on Λ is given by an assignment of a spin j_f to every face f of Λ such that at every interior edge e of Λ the triangle inequality is satisfied by the three adjacent spins. Dually, the spin foam can be described as a configuration on the dual complex Λ^* : then, the spin foam F is specified by spins j_e on edges of Λ^* , where for every triangle of Λ^* , the spins on the edges of the triangle satisfy the triangle inequality.

We define a spin network S on the boundary $\partial\Lambda$ as an assignment of spins j_e to edges in the boundary $\partial\Lambda$ such that for every vertex in the boundary the adjacent spins satisfy the triangle inequality. A particularly simple example of a spin network is a non-selfintersecting loop C that carries a spin label j . We denote such a spin network by (C, j) . Each spin foam on Λ induces a spin network on the boundary $\partial\Lambda$, which we call the boundary ∂F of F .

III. SU(2) LATTICE YANG-MILLS THEORY IN 3 DIMENSIONS

The partition function of 3-dimensional SU(2) lattice Yang–Mills theory is defined by a path integral over SU(2)-valued link (or edge) variables U_e on a cubic lattice κ :

$$Z = \int \left(\prod_{e \in \kappa} dU_e \right) \exp \left(- \sum_f \mathcal{S}_f(U_f) \right) \quad (1)$$

The face (or plaquette) action \mathcal{S}_f depends on the holonomy U_f around the face. As in paper I, we choose \mathcal{S}_f to be the heat kernel action (for more details on the definition, see [34]). The heat kernel action has a particularly simple expansion in terms of characters, namely,

$$\exp \left(- \mathcal{S}_f(U_f) \right) = \sum_j (2j+1) e^{-\frac{2}{\beta} j(j+1)} \chi_j(U_f). \quad (2)$$

The coupling factor β is related to the gauge coupling g via

$$\beta = \frac{4}{ag^2} + \frac{1}{3}. \quad (3)$$

The expectation value of a Wilson loop C in the representation j is

$$\langle \text{tr}_j U_C \rangle = \int \left(\prod_{e \in \kappa} dU_e \right) \text{tr}_j U_C \exp \left(- \sum_f \mathcal{S}_f(U_f) \right). \quad (4)$$

U_C denotes the holonomy along the loop C .

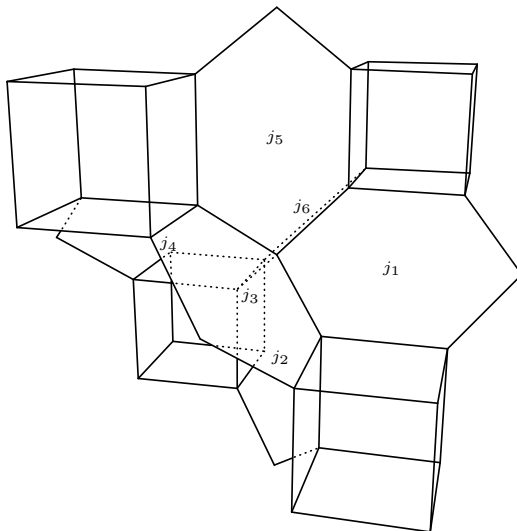


Figure 1: Tessellation of \mathbb{R}^3 by cubes and truncated rhombic dodecahedra.

IV. SPIN FOAM REPRESENTATION

A. Partition function

In general, there are several, equivalent ways of writing down a sum over spin foams. Here, we will use a scheme by Anishetty, Cheluvaraja, Sharatchandra and Mathur [27], where the amplitude is expressed in terms of $6j$ -symbols¹. In the paper by Anishetty et al., spin foams are described by spin assignments j_e to *edges* of a triangulation T . For the purpose of the present paper, it is convenient to go to the dual picture where spin foams are spin assignments j_f to *faces* of the dual T^* . Let us call this lattice $\tilde{\kappa}$. It is given by a tessellation of the 3-dimensional space by cubes and truncated rhombic dodecahedra (see Fig. 1).

The complex $\tilde{\kappa}$ contains two types of faces: square faces that correspond to faces of the original cubic lattice κ , and hexagonal faces that connect pairs of square faces. At each edge of $\tilde{\kappa}$, exactly three faces meet, and at each vertex we have six intersecting faces. We will be slightly sloppy with our notation and write $f \subset \kappa$ to denote the square faces of $\tilde{\kappa}$.

After the dual transformation, the partition function (1) is expressed as a sum over closed spin foams F on $\tilde{\kappa}$, where each spin foam carries a certain weight:

$$Z = \sum_{F \mid \partial F = \emptyset} \left(\prod_{f \subset \tilde{\kappa}} (2j_f + 1) \right) \left(\prod_{v \subset \tilde{\kappa}} A_v \right) \left(\prod_{f \subset \kappa} (-1)^{2j_f} e^{-\frac{2}{\beta} j_f(j_f+1)} \right). \quad (5)$$

In the amplitude, every face contributes with the dimension $2j_f + 1$ of the representation j_f . In addition, square faces give an exponential of the Casimir and a sign factor $(-1)^{2j_f}$. For each vertex of $\tilde{\kappa}$, we get the value of a so-called tetrahedral spin network as a factor:

$$A_v = \begin{array}{c} \text{Diagram of a tetrahedron with vertices labeled } j_1, j_2, j_3, j_4 \text{ on the edges and } j_5, j_6 \text{ on the faces.} \end{array} = \left\{ \begin{array}{ccc} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{array} \right\} \quad (6)$$

¹ Recently, the same result was obtained very efficiently by the use of Kauffman–Lins spin networks [35].

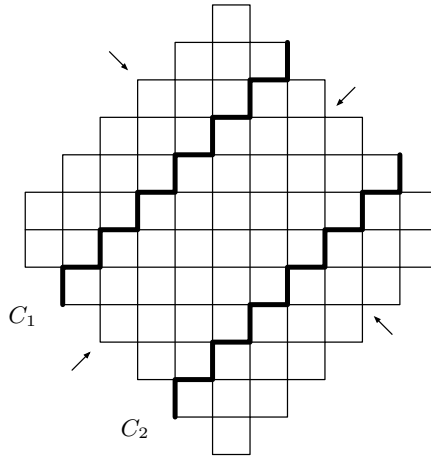


Figure 2: Zig-zag path of the Polyakov loops C_1 and C_2 in a 2d slice of the lattice κ . The arrows indicate how lattice points are identified.

The edges of the tetrahedral spin network correspond to faces of the spin foam surrounding the vertex v , and the vertices of the spin network correspond to the edges where these faces meet (see Fig. 1). The value of the spin network is equal to a $6j$ -symbol, where the spins j_1 , j_2 and j_3 are read off from any vertex of the tetrahedron.

B. Polyakov loops

The dual transformation can be also applied to expectation values of observables such as Wilson loops or products of them. When the dual transform of such loops is computed, the explicit form of amplitudes depends on the geometry of the loops. For a rectangular Wilson loop, it was explicitly determined by Diakonov & Petrov [28]. In ref. [36], one of us derived the dual amplitude for Polyakov loops. In the following, we will consider the example of Polyakov loops, since everywhere along the loops the amplitude has the same structure. In the case of a rectangular Wilson loop, one has to distinguish between the straight part and the corners of the loop.

We let the Polyakov loops C_1 and C_2 run along zig-zag paths through the lattice κ and adopt boundary conditions that identify lattice points on opposing ends of diagonals² (see Fig. 2). As before, we introduce a tessellation $\tilde{\kappa}$, where square faces correspond to faces of the original lattice, and hexagonal faces connect pairs of such faces. To describe the spin foam sum for the Polyakov loops, we need to modify this lattice. This happens in several steps: first we remove all 3-cells, so that we obtain the 2-skeleton of $\tilde{\kappa}$. In $\tilde{\kappa}$ the Polyakov loops C_1 and C_2 correspond to two closed sequences of hexagons. Imagine that we draw a closed loop within each sequence that connects the centers of neighbouring hexagons (see Fig. 3). For each pair of neighbouring hexagons, we also add an edge that connects their centers directly, i.e. in a straight line outside the 2-complex. Each such edge forms a triangle with the edges inside the hexagons. We include these triangular faces in the complex, and call the resulting 2-complex again $\tilde{\kappa}$. Its boundary consists of two loops which we denote by \tilde{C}_1 and \tilde{C}_2 respectively.

Using this complex, we can describe the spin foam sum of the two Polyakov loops as follows. It is given by

$$\langle \text{tr}_j U_{C_1} \text{tr}_j U_{C_2} \rangle = \frac{1}{Z} \sum_{F \mid \partial F = (\tilde{C}_1 \cup \tilde{C}_2, j)} \left(\prod_{f \in \tilde{\kappa}} (2j_f + 1) \right) \left(\prod_{v \in \tilde{\kappa}} A_v \right) \left(\prod_{f \in \kappa} (-1)^{2j_f} e^{-\frac{2}{\beta} j_f(j_f+1)} \right). \quad (7)$$

The difference to (5) consists of the modification of the complex and the boundary condition $\partial F = (\tilde{C}_1 \cup \tilde{C}_2, j)$. The boundary condition $\partial F = (\tilde{C}_1 \cup \tilde{C}_2, j)$ requires that the spin on the loop edges is j . The attachment of triangles along $\tilde{C}_1 \cup \tilde{C}_2$ creates two types of new vertices in the complex: vertices in the middle of hexagons along $\tilde{C}_1 \cup \tilde{C}_2$, and

² The use of zig-zag paths is not essential for the result of this paper. We choose these paths for convenience, since in this case the amplitudes are already known from ref. [36].

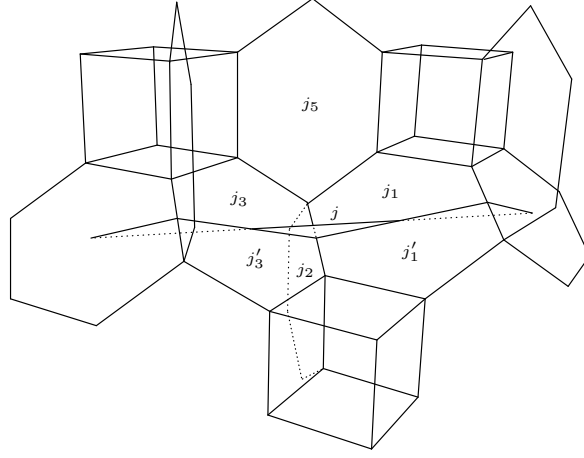


Figure 3: Modification of the complex $\tilde{\kappa}$: the effect of the Polyakov loops can be described by inserting additional faces.

vertices in the middle of the boundary edge between such hexagons. In the first case, the vertex amplitude is trivial, i.e.

$$A_v = 1. \quad (8)$$

To the second type of vertex we associate a tetrahedral spin network whose edges and vertices correspond to faces and edges around this vertex:

$$A_v = (-1)^{j_3-j'_3} (-1)^{j_1-j'_1} (-1)^{j_1+j_3+j_2+j} \quad (9)$$

$$= (-1)^{j_3-j'_3} (-1)^{j_1-j'_1} (-1)^{j_1+j_3+j_2+j} \left\{ \begin{matrix} j_1 & j_3 & j_2 \\ j'_3 & j'_1 & j \end{matrix} \right\}. \quad (10)$$

The spins j_1 , j_2 and j_3 are read off from one of the two vertices not adjacent to j : if the edge with spin j is drawn at the top (as in Fig. 3), this vertex is on the left side of j in the direction of passage of the Polyakov loop, i.e. on the left side in the direction from j_3 , j'_3 towards j_1 , j'_1 .

V. WORLD SHEET INTERPRETATION OF SPIN FOAMS

A. Definition of worldsheets

To arrive at the worldsheet interpretation of spin foams, we have to apply a further modification to the complex $\tilde{\kappa}$. We “frame” $\tilde{\kappa}$, so that it becomes a 3-complex. Under this framing each 2-cell f of $\tilde{\kappa}$ is turned into a 3-cell f' that has the topology of $f \times (0, 1)$. Neighbouring cells are connected as in Fig. 4 and Fig. 6a. The resulting 3-complex is called κ' . The precise metric properties of κ' do not matter as long as it has the required cell structure. The framing of $\tilde{\kappa}$ induces also a framing of the boundary $\tilde{\kappa}$. Each 1-cell $e \subset \partial \tilde{\kappa}$ is thickened into a 2-cell e' that has the topology of a half-open strip $[0, 1] \times (0, 1)$. Note that the boundary $\partial e'$ of e' is disconnected. When we speak of the boundary $\partial \kappa'$ of κ' , we mean the union of all such framed edges e' : they form two ribbons—the framed version of the two loops \tilde{C}_1 and \tilde{C}_2 (see Fig. 3).

Consider a compact embedded surface³ S in κ' whose boundary lies in $\partial \kappa'$. Take a framed 3-cell f' in κ' and determine the intersection $S \cap \partial f'$ of the surface with the cell boundary $\partial f'$. In general, this intersection can be

³ The embedding implies, in particular, that surface does not intersect with itself.

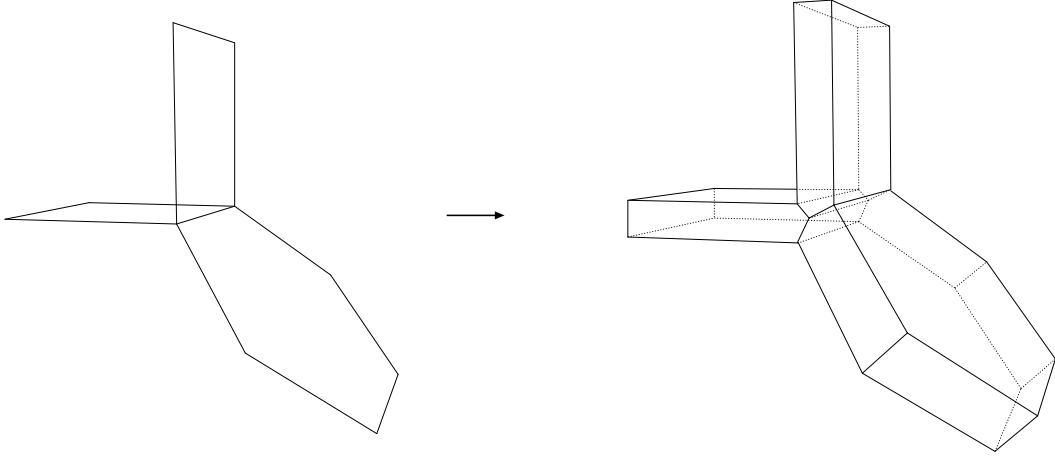


Figure 4: Under the framing three faces of $\tilde{\kappa}$ along an edge become three 3-cells that intersect along 2-cells.

empty or consist of loops, lines and points. The cell boundary $\partial f'$ has the topology of an open annulus, so there are two types of loops: loops that are contractible in $\partial f'$ and loops that are not. Let us assume that for any $f' \subset \kappa'$, the intersection $S \cap \partial f'$ contains only loops of the non-contractible kind. We count the number of such loops in $\partial f'$ and call it N_f . Obviously, this number does not change if we apply a homeomorphism to S that is connected to the identity and maps cell boundaries $\partial f'$ onto themselves. In this limited sense, the numbers N_f , $f \subset \tilde{\kappa}$, are topological invariants.

Moreover, they satisfy constraints. To see this, consider a triple f_1, f_2, f_3 of faces that intersect along an edge e of $\tilde{\kappa}$. Correspondingly, we have three framed faces f'_1, f'_2, f'_3 of κ' that intersect along 2-cells $e'_{12}, e'_{23}, e'_{31}$ (see Fig. 4). The surface $S \subset \kappa'$ induces non-contractible loops within the boundaries $\partial f'_1, \partial f'_2, \partial f'_3$ (see Fig. 5). Clearly, each loop in a boundary $\partial f'_i$ borders exactly one loop from another boundary $\partial f'_j$, $i \neq j$. This pairing of loops implies that the numbers $N_{f_1}, N_{f_2}, N_{f_3}$ satisfy the triangle inequality

$$|N_{f_1} - N_{f_2}| \leq N_{f_3} \leq N_{f_1} + N_{f_2}. \quad (11)$$

If we write $j_f = N_f/2$, this is precisely the spin coupling constraint that defines a spin foam F with spins j_f . We see therefore that the numbers N_f define spin foams F on $\tilde{\kappa}$! We will show, in fact, that for every spin foam F there is a surface S whose loop numbers are given by F , and if we restrict the surfaces suitably there is a bijection between surfaces in κ' and spin foams on $\tilde{\kappa}$. On the boundary this relation induces a correspondence between curves on $\partial\kappa'$ and spin networks on $\partial\tilde{\kappa}$.

We will first define a suitable class of surfaces and curves, and then prove that the bijection holds. Motivated by the well-known conjectures about gauge-string dualities, we call the surfaces and curves worldsheets and strings. Equivalence relations will be furnished by homeomorphisms $h : \Lambda \rightarrow \Lambda$ on n -complexes Λ , $n = 2, 3$, that

1. map boundaries ∂c of n -cells c onto themselves, and
2. are connected to the identity through homeomorphisms with property 1.

Let $\text{Homeo}(\Lambda)$ denote the set of such restricted homeomorphisms.

Definition A string γ on κ' is an embedded, not necessarily connected, compact closed curve in the boundary of κ' where for each 2-cell c of $\partial\kappa'$ the intersection $\gamma \cap c$ consists of lines and each line intersects ∂c in two end points that are not contractible in ∂c .

We consider two strings γ and γ' as equivalent if they are related by a homeomorphism $h \in \text{Homeo}(\partial\kappa')$.

Definition A worldsheet w on κ' is an embedded, not necessarily connected, compact surface in κ' such that

- (i) the boundary ∂w is a string on $\partial\kappa'$, and
- (ii) for each 3-cell f' of κ' the intersection $w \cap f'$ consists of disks and each disk intersects $\partial f'$ in a loop that is non-contractible in $\partial f'$.

We consider two worldsheets w and w' as equivalent if they are related by a homeomorphism $h \in \text{Homeo}(\kappa')$.

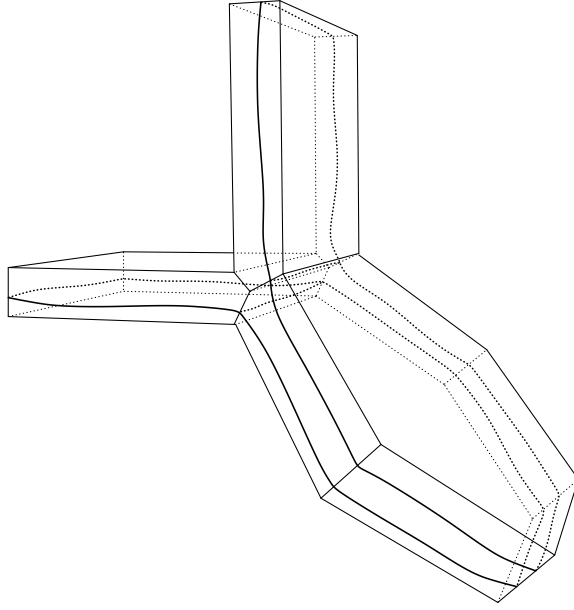


Figure 5: A surface S induces loops in the boundary of 3-cells of κ' .

B. Correspondence between spin foams and worldsheets

Since the boundary of κ' has the topology of $S^1 \cup S^1$, the correspondence between strings on $\partial\kappa'$ and spin networks on $\partial\tilde{\kappa}$ is rather trivial. It is clear from the definition that a string on $\partial\kappa'$ is a union of N_1 disjoint loops along $\tilde{C}_1 \times (0, 1)$ and N_2 disjoint loops along $\tilde{C}_2 \times (0, 1)$. We denote this string by $\gamma_{\tilde{C}_1, N_1} \cup \gamma_{\tilde{C}_2, N_2}$. On the other hand, the only possible spin networks are given by the loops $(\tilde{C}_1, j_1) \cup (\tilde{C}_2, j_2)$ with spin j_1 and j_2 . Therefore, a one-to-one correspondence is set up by associating the string $\gamma_{\tilde{C}_1, 2j_1} \cup \gamma_{\tilde{C}_2, 2j_2}$ to the spin network $(\tilde{C}_1, j_1) \cup (\tilde{C}_2, j_2)$.

Let us now choose fixed values for the spins j_1 and j_2 . Denote the set of all spin foams F s.t. $\partial F = (\tilde{C}_1, j_1) \cup (\tilde{C}_2, j_2)$ by \mathcal{F} , and let W stand for the set of worldsheets s.t. $\partial w = \gamma_{\tilde{C}_1, 2j_1} \cup \gamma_{\tilde{C}_2, 2j_2}$.

Proposition V.1 *There is a bijection $f : \mathcal{F} \rightarrow W$ between spin foams F on $\tilde{\kappa}$ s.t. $\partial F = (\tilde{C}_1, j_1) \cup (\tilde{C}_2, j_2)$ and worldsheets w on κ' s.t. $\partial w = \gamma_{\tilde{C}_1, 2j_1} \cup \gamma_{\tilde{C}_2, 2j_2}$.*

Proof We start by constructing a map $f : \mathcal{F} \rightarrow W$. Then, we will show that f is injective and surjective.

Let F be a spin foam in \mathcal{F} . Consider the vertices v of κ' where six 3-cells intersect. Denote the set of these vertices as V' . Around each vertex $v \in V'$ we choose a closed ball $B_\epsilon(v)$ of radius ϵ . The intersection of the balls with cells of κ' defines a new, finer complex that we call κ'_\pm . We can view this complex as the union of two complexes κ'_+ and κ'_- , where κ'_+ results from κ'_\pm by removing the interior of all balls $B_\epsilon(v)$:

$$\kappa'_+ = \kappa'_\pm \setminus \bigcup_{v \in V'} B_\epsilon^\circ(v) \quad (12)$$

κ'_- , on the other hand, is the subcomplex of κ'_\pm that remains when we keep the balls $B_\epsilon(v)$ and delete the rest. Every 3-cell f' of κ' is a union

$$f' = f'_+ \cup \bigcup_i f'_{-i} \quad (13)$$

where f'_+ is a 3-cell of κ'_+ and the f'_{-i} , $i = 1, \dots, n$, are 3-cells in κ'_- .

In order to construct the worldsheet corresponding to the spin foam F , we will first build a surface in the complex κ'_+ . In the second step, we will also fill the balls $B_\epsilon(v)$ with surfaces, so that the union of all surfaces gives a worldsheet in κ' . Consider an arbitrary face f of $\tilde{\kappa}$ with spin j_f determined by the spin foam F . The corresponding 3-cell f'_+ in κ'_+ has the topology of a closed 3-ball with two punctures. Its boundary $\partial f'_+$ is an open annulus. In each such 3-cell f'_+ we place $N_f = 2j_f$ disjoint closed disks whose boundary is given by non-contractible loops in $\partial f'_+$. Along every

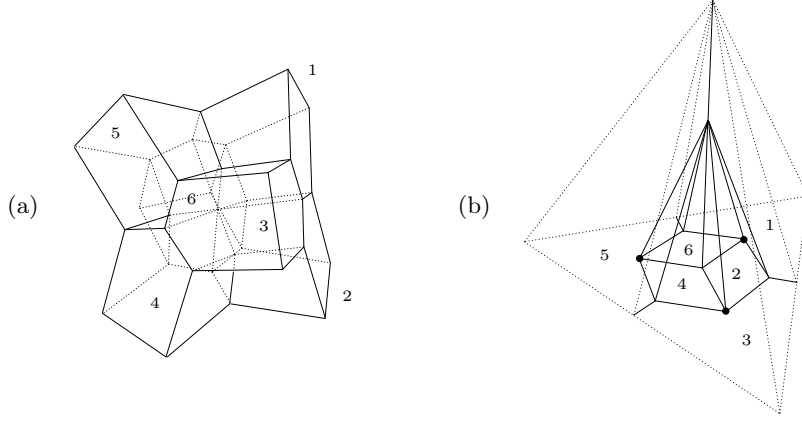


Figure 6: (a) A closed ball $B_\epsilon(v)$ in κ' around a vertex v where six framed cells meet. The cells of κ' induce a cell structure in the ball. The resulting cell complex is topologically equivalent to the complex in Fig. 6b. (b) A tetrahedron in \mathbb{R}^3 with an open triangle at the bottom, and triangles removed on the three other sides. The boundary of the ball $B_\epsilon(v)$ is mapped onto the bottom triangle. Solid lines delineate the boundaries between 3-cells in the interior of the tetrahedron. The three thick dots indicate punctures. The three missing triangles in the boundary form a fourth puncture.

edge e in the interior of κ'_+ three 3-cells $f'_{+1}, f'_{+2}, f'_{+3}$ intersect. Due to the spin coupling conditions, the numbers $N_{f'_{+1}}, N_{f'_{+2}}, N_{f'_{+3}}$ satisfy the triangle inequality

$$|N_{f'_{+1}} - N_{f'_{+2}}| \leq N_{f'_{+3}} \leq N_{f'_{+1}} + N_{f'_{+2}}. \quad (14)$$

This implies that we can rearrange the disks in such a way that their boundary edges are pairwise congruent at the shared boundaries of the cells $f'_{+1}, f'_{+2}, f'_{+3}$. We repeat this procedure for every edge $e \in \kappa'_+$ where three 3-cells meet, and thereby obtain a compact embedded surface w_+ in κ'_+ . Up to homeomorphisms $h \in \text{Homeo}(\kappa'_+)$, this surface is uniquely determined by our procedure.

We now explain how we fill the “holes” in κ'_+ , so that we get a surface in the entire complex κ' . Each ball $B_\epsilon(v)$ defines a subcomplex of κ'_- as depicted in Fig. 6a. It consists of six 3-cells c_1, \dots, c_6 . The boundary ∂c_i of each cell is topologically an open annulus, and subdivided into five 2-cells. Four of these 2-cells are shared with neighbouring 3-cells c_j , $j \neq i$, and one of them lies in the boundary $\partial B_\epsilon(v)$ of the ball. We call the former type of 2-cell internal, and the latter one external.

To fill this complex with surfaces, it is helpful to use another, topologically equivalent complex that is shown in Fig. 6b: the interior of the ball $B_\epsilon(v)$ corresponds to the interior of a tetrahedron and the boundary $\partial B_\epsilon(v)$ is projected onto one of the four triangles. This triangle has three punctures.

For every ball $B_\epsilon(v)$, the surface w_+ induces an embedded closed curve γ_v along the boundary $\partial B_\epsilon(v)$. The curve consists of n loops l_i , i.e. $\gamma_v = l_1 \cup \dots \cup l_n$. In the alternative representation of Fig. 6b, the curve appears as a set of embedded loops in the bottom triangle that wind around the three punctures (see Fig. 7). To create the surface in $B_\epsilon(v)$, we will cover the n loops by n disks in $B_\epsilon(v)$. This will be done in such a way that condition (ii) for worldsheets is satisfied.

Consider a single 3-cell c_i in the ball $B_\epsilon(v)$, and the one external 2-cell in its boundary ∂c_i . The intersection of the curve γ_v with this 2-cell gives a number of lines e_{ik} , $k = 1, \dots, K_i$. Each of the two end points of a line e_{ik} is located on a 1-cell shared by an external and an internal 2-cell of ∂c_i . Let us now draw a line from one of the end points through the internal 2-cell to the vertex in the center of $B_\epsilon(v)$. The same is done for the second end point. Together with the original line e_{ik} , these lines form a loop in the cell boundary ∂c_i . We fill this loop with a disk d_{ik} in c_i , so that the intersection $d_{ik} \cap c_i$ is again the loop.

We repeat this procedure for every line e_{ik} in the cell c_i , and then in every cell c_i . Along the boundary between neighbouring 3-cells, we glue the disks together: when a line e_{ik} is connected to another line $e_{i'k'}$, $i \neq i'$, the corresponding disks are glued together along the internal 2-cell $\partial c_i \cap \partial c_{i'}$. This can be done in such way that the resulting total surface intersects only in one point: at the vertex in the center of $B_\epsilon(v)$, like a stack of sheets that are pinched together. Let us call this surface w_- .

Observe that w_- satisfies property (ii) in the subcomplex $B_\epsilon(v)$. Due to the way we have placed disks outside of $B_\epsilon(v)$, every line e_{ik} connects 1-cells of $\partial B_\epsilon(v)$ that are disconnected. As a result, each loop $d_{ik} \cap c_i$ is non-contractible in ∂c_i .

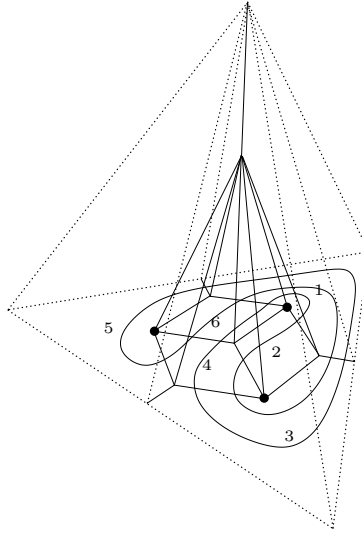


Figure 7: Example of an induced loop in the boundary of the ball $B_\epsilon(v)$.

To arrive at an embedded surface, we need to remove the point of degeneracy at the center of the ball $B_\epsilon(v)$. We do so by moving the different parts of w_- slightly apart, and in such a way that no new components are created in the intersections $w_- \cap c_i$. The latter ensures that the new surface w_- still has property (ii). Up to homeomorphisms $h \in \text{Homeo}(B_\epsilon(v))$ which leave γ_v invariant, w_- is the unique embedded surface that is bounded by γ_v and meets condition (ii).

After filling each ball $B_\epsilon(v)$ with such a surface w_- , we take the union of the surfaces w_- with w_+ . This gives us an embedded compact surface w in κ' . Let us check if w meets requirement (i) and (ii) of the definition of a worldsheet.

Due to the arrangement of disks in 3-cells c of κ'_+ , the induced loops in the boundary ∂c never connect 1-cells that are connected in ∂c . This means, in particular, that the induced curve in the boundary $\partial \kappa'$ consists of lines in 2-cells c , where each line connects two disconnected 1-cells of ∂c . Therefore, the boundary of each line cannot be contracted in ∂c , and the surface w has property (i).

How about property (ii)? The surface has the desired property for the cells of κ'_+ , and we showed the same for each subcomplex $B_\epsilon(v)$. It is clear from this that w has property (ii) in κ' .

We conclude that w is a worldsheet on κ' . The whole construction defines a map $f : \mathcal{F} \rightarrow W$ from spin foams to worldsheets.

Next we prove that f is injective and surjective. Let F and F' be two different spin foams. There must be a face $f \subset \tilde{\kappa}$ for which $N_f \neq N'_f$. This implies that the corresponding worldsheets w and w' are different, since they have different invariants under the homeomorphisms $h \in \text{Homeo}(\kappa')$. Thus, f is injective.

To check surjectivity, consider an arbitrary worldsheet $w \in W$. Within each 3-cell c of κ' , the worldsheet induces N_f disks that are bounded by non-contractible loops in ∂c . The numbers N_f define a spin foam F with spins $j_f = N_f/2$. From F we construct another worldsheet $w' = f(F)$. Provided the balls $B_\epsilon(v)$ are chosen small enough, the intersections $w \cap \kappa^+$ and $w' \cap \kappa^+$ are related by a homeomorphism $h \in \text{Homeo}(\kappa'_+)$. Inside the balls $B_\epsilon(v)$, the worldsheet w' consists of disks that are bounded by loops in $\partial B_\epsilon(v)$. Up to homeomorphisms $h_v \in \text{Homeo}(B_\epsilon(v))$ that leave γ_v invariant, there is precisely one way to cover these loops by disks in $B_\epsilon(v)$ such that property (ii) is met. For sufficiently small ϵ , the intersection $w \cap B_\epsilon(v)$ has property (ii) as well, and must be related to $w' \cap B_\epsilon(v)$ by a homeomorphism $h_v \in \text{Homeo}(B_\epsilon(v))$. Thus, there is a homeomorphism $h \in \text{Homeo}(\kappa')$ that relates w and w' , and $w' = f(F) = w$. This shows that f is surjective and completes the proof.

VI. STRING REPRESENTATION OF 3D SU(2) LATTICE YANG-MILLS THEORY

By using the correspondence between spin foams and worldsheets, we can now translate the exact dual representations (5) and (7) into exact string representations of 3d SU(2) Yang-Mills theory. The string representation is defined on a complex κ' that arises from a framing of the 2-skeleton of a tessellation $\tilde{\kappa}$ by cubes and truncated rhombic dodecahedra (see Fig. 1, Fig. 4 and Fig. 6a). Under the framing, faces f of the 2-skeleton become 3-cells f' of the framed complex κ' . Vertices v turn into vertices $v' \subset \kappa'$, where six framed 3-cells f' intersect. The set of these

vertices v' is denoted by V' . The 3-cells f' of κ' belong to two groups: 3-cells f' that originate from square faces f of $\tilde{\kappa}$ (and correspond to faces $f \in \kappa$), and those arising from hexagonal faces in $\tilde{\kappa}$. Worldsheets and strings are defined as certain surfaces and curves in the framed complex κ' (see sec. V).

With these conventions, the partition function is given by a sum over closed worldsheets:

$$Z = \sum_{w \mid \partial w = \emptyset} \left(\prod_{f' \subset \kappa'} (N_{f'} + 1) \right) \left(\prod_{v' \subset V'} A_{v'}(\{N_{f'}/2\}) \right) \left(\prod_{f \subset \kappa} (-1)^{N_{f'}} e^{-\frac{1}{2\beta} N_{f'}(N_{f'}+2)} \right) \quad (15)$$

The amplitude has three contributions: every framed face contributes with a factor $N_{f'} + 1$, where $N_{f'}$ is the number of components of the worldsheet in f' . In addition, square faces give an exponential and a sign factor $(-1)^{N_{f'}}$. For each vertex $v \in V'$, we get a $6j$ -symbol

$$A_{v'}(\{N_{f'}/2\}) = \left\{ \begin{matrix} N_{f'_1}/2 & N_{f'_2}/2 & N_{f'_3}/2 \\ N_{f'_4}/2 & N_{f'_5}/2 & N_{f'_6}/2 \end{matrix} \right\} \quad (16)$$

where the f'_i are the six 3-cells that intersect at v' .

For the expectation value of two Polyakov loops (C_1, j) and (C_2, j) (as defined in section sec. IV B), an additional modification of the 2-skeleton was required: we insert a sequence of triangles along two loops \tilde{C}_1 and \tilde{C}_2 (see Fig. 3). Under the framing, the two loops become ribbons. The expectation value of the Polyakov loops is equal to a sum over worldsheets that are bounded by $2j$ strings along the first ribbon $\tilde{C}_1 \times (0, 1)$ and by $2j$ strings along the second ribbon $\tilde{C}_2 \times (0, 1)$. Denoting these strings as $\gamma_{\tilde{C}_1, 2j} \cup \gamma_{\tilde{C}_2, 2j}$, the sum takes the form

$$\langle \text{tr}_j U_{C_1} \text{tr}_j U_{C_2} \rangle = \sum_{w \mid \partial w = \gamma_{\tilde{C}_1, 2j} \cup \gamma_{\tilde{C}_2, 2j}} \left(\prod_{f' \subset \kappa'} (N_{f'} + 1) \right) \left(\prod_{v' \subset V'} A_{v'}(\{N_{f'}/2\}) \right) \left(\prod_{f \subset \kappa} (-1)^{N_{f'}} e^{-\frac{1}{2\beta} N_{f'}(N_{f'}+2)} \right). \quad (17)$$

The difference to (5) consists of the modification of the complex and the boundary condition $\partial w = \gamma_{\tilde{C}_1, 2j} \cup \gamma_{\tilde{C}_2, 2j}$. The attachment of triangles along $\tilde{C}_1 \cup \tilde{C}_2$ creates two types of new vertices in κ' : vertices in the middle of framed hexagons along the ribbons, and vertices in the middle of the boundary between such hexagons. In the first case, the vertex amplitude is trivial, i.e.

$$A_{v'} = 1. \quad (18)$$

To the second type of vertex we attribute a factor

$$A_{v'} = (-1)^{(N_{f'_3} - N'_{f'_3})/2} (-1)^{(N_{f'_1} - N'_{f'_1})/2} (-1)^{(N_{f'_1} + N_{f'_3} + N_{f'_2} + 2j)/2} \left\{ \begin{matrix} N_{f'_1}/2 & N_{f'_3}/2 & N_{f'_2}/2 \\ N'_{f'_3}/2 & N'_{f'_1}/2 & j \end{matrix} \right\} \quad (19)$$

The labelling is analogous to the labelling by spins in eq. (10).

In this string representation, N -ality dependence and string breaking take on a very concrete form. For spin $j = 1/2$, the boundary string consists of two loops $\gamma_{\tilde{C}_1, 1}$ and $\gamma_{\tilde{C}_2, 1}$: one along the ribbon $\tilde{C}_1 \times (0, 1)$ and the other one along the ribbon $\tilde{C}_2 \times (0, 1)$. Since every worldsheet has to be bounded by the string $\gamma_{\tilde{C}_1, 1} \cup \gamma_{\tilde{C}_2, 1}$, there is necessarily a connected component of the worldsheet that connects the boundary strings $\gamma_{\tilde{C}_1, 1}$ and $\gamma_{\tilde{C}_2, 1}$. The string between quarks is “unbroken”. When we go to $j = 1$, on the other hand, we have a pair $\gamma_{\tilde{C}_1, 2}$ of strings along $\tilde{C}_1 \times (0, 1)$ and a pair $\gamma_{\tilde{C}_2, 2}$ of strings along $\tilde{C}_2 \times (0, 1)$. In this case, the four single strings can be either connected by two surfaces that go across the distance between the Polyakov loops, or each pair is connected to itself by a tube-like surface. In the latter case, the string between quarks is “broken”. As we go to higher spins, the worldsheet can consist of several extended surfaces, several tube-like surfaces or a mixture of both.

VII. DISCUSSION

In this paper, we showed that 3d SU(2) lattice Yang–Mills theory can be cast in the form of an exact string representation. Our starting point was the exact dual (or spin foam) representation of the lattice gauge theory. We demonstrated that spin foams can be equivalently described as self-avoiding worldsheets of strings on a framed lattice. This lattice arose in two steps: we replaced the original cubic lattice by a tessellation, where at every edge only three

faces intersect. Then, we took the 2-skeleton of this complex, and framed (or thickened) it by choosing an open neighbourhood of it in \mathbb{R}^3 . We proved that there is a bijection between a subset of surfaces in the framed complex and spin foams in the unframed complex. This allowed us to translate the partition function from a sum over spin foams into a sum over closed worldsheets. The expectation value of two Polyakov loops with spin j became a sum over worldsheets that are bounded by $2j$ strings along each loop.

To our knowledge, this is the first example of an exact and fully explicit string representation of $SU(2)$ lattice Yang–Mills theory in three dimensions⁴. Not surprisingly, it differs from a simple Nambu–Goto string. When a worldsheet does not run more than once through faces (i.e. when $N_{f'} \leq 1$), the $6j$ -symbols in the amplitude become trivial and the exponent in (15) is proportional to the area of the worldsheet. In these cases, the weighting resembles that of the Nambu–Goto string. In general, however, a worldsheet intersects several times with the same cell, and then we have an interaction due to nonlinear dependences on $N_{f'}$. That is, in addition to interactions by merging and splitting, there is an interaction of directly neighbouring strings. Note that this does not preclude the possibility that a Nambu–Goto string gives a good effective description in special cases or regimes.

It is interesting to compare this result to the AdS–CFT correspondence, where the gauge–string duality is constructed by completely different methods. One should also observe the difference between our “non-abelian” worldsheets and the surfaces that arise in abelian lattice gauge theory. In the case of $U(1)$, the theory can be transformed to a sum over closed 2-chains, and in this sense one has a sum over surfaces. The worldsheets of our string representation are of the same type as long as $N_{f'} \leq 1$. When the occupation number increases, however, the surfaces can be “jammed” against each other along faces without being “added” like abelian 2-chains.

At a practical level, the present worldsheet picture could be useful for analyzing the dual representation. It could be helpful, for example, when thinking about “large” moves in Monte Carlo simulations [35]: by inserting an entire worldsheet into a given worldsheet, one can create a non-local change in spin foams that is compatible with the spin-coupling conditions.

A possible shortcoming of the present work is the restriction on the shape of surfaces. It was needed in order to establish the bijection between worldsheets and spin foams. From a mathematical perspective, it would be more elegant to admit arbitrary compact self-avoiding surfaces, and to characterize spin foams as certain topological invariants. We hope to obtain such a characterization in future work.

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- [1] A.M. Polyakov. *Gauge fields and strings*. Harwood, 1987.
 - [2] A.M. Polyakov. Confinement and liberation. 2004, hep-th/0407209.
 - [3] A.M. Polyakov. Confining strings. *Nucl. Phys.*, B486:23–33, 1997, hep-th/9607049.
 - [4] D. Antonov. String representation and nonperturbative properties of gauge theories. *Surveys High Energ. Phys.*, 14:265–355, 2000, hep-th/9909209.
 - [5] T. Eguchi and K. Nishijima, editors. *Broken symmetries: selected papers of Y. Nambu*. World Scientific, Singapore, 1995.
 - [6] L. Susskind. Dual symmetric theory of hadrons. 1. *Nuovo Cim.*, A69:457–496, 1970.
 - [7] G. Frye, C.W. Lee, and L. Susskind. Dual-symmetric theory of hadrons. 2. baryons. *Nuovo Cim.*, A69:497–507, 1970.
 - [8] D.B. Fairlie and H.B. Nielsen. An analog model for KSV theory. *Nucl. Phys.*, B20:637–651, 1970.
 - [9] J.H. Schwarz. String theory: The early years. 2000, hep-th/0007118.
 - [10] K.G. Wilson. Confinement of quarks. *Phys. Rev.*, D10:2445–2459, 1974.
 - [11] A.M. Polyakov. Gauge fields as rings of glue. *Nucl. Phys.*, B164:171–188, 1980.
 - [12] A.A. Migdal. QCD: Fermi string theory. *Nucl. Phys.*, B189:253, 1981.
 - [13] M. Lüscher, K. Symanzik, and P. Weisz. Anomalies of the free loop wave equation in the WKB approximation. *Nucl. Phys.*, B173:365, 1980.
 - [14] O. Alvarez. The static potential in string models. *Phys. Rev.*, D24:440, 1981.
 - [15] J.F. Arvis. The exact q anti-q potential in Nambu string theory. *Phys. Lett.*, B127:106, 1983.
 - [16] J. Ambjorn, B. Durhuus, and T. Jonsson. *Quantum geometry. A statistical field theory approach*. Cambridge Monogr. Math. Phys., Cambridge, 1997.

⁴ In the case of 2d QCD, an exact string representation was found by Gross and Taylor [37, 38].

- [17] L. Brink, P. Di Vecchia, and P.S. Howe. A locally supersymmetric and reparametrization invariant action for the spinning string. *Phys. Lett.*, B65:471–474, 1976.
- [18] L. Brink, P. Di Vecchia, and P.S. Howe. A Lagrangian formulation of the classical and quantum dynamics of spinning particles. *Nucl. Phys.*, B118:76, 1977.
- [19] A.M. Polyakov. Quantum geometry of bosonic strings. *Phys. Lett.*, B103:207–210, 1981.
- [20] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys.*, B72:461, 1974.
- [21] J.M. Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998, hep-th/9711200.
- [22] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov. Gauge theory correlators from non-critical string theory. *Phys. Lett.*, B428:105–114, 1998, hep-th/9802109.
- [23] E. Witten. Anti-de sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998, hep-th/9802150.
- [24] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, and Y. Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000, hep-th/9905111.
- [25] G. Münster. High temperature expansions for the free energy of vortices, respectively the string tension in lattice gauge theories. *Nucl. Phys.*, B180:23, 1981.
- [26] J.-M. Drouffe and J.-B. Zuber. Strong coupling and mean field methods in lattice gauge theories. *Phys. Rept.*, 102:1, 1983.
- [27] R. Anishetty, S. Cheluvarama, H.S. Sharatchandra, and M. Mathur. Dual of three-dimensional pure SU(2) lattice gauge theory and the Ponzano-Regge model. *Phys. Lett.*, B314:387–390, 1993, hep-lat/9210024.
- [28] D. Diakonov and V. Petrov. Yang-Mills theory in three dimensions as quantum gravity theory. *J. Exp. Theor. Phys.*, 91:873–893, 2000, hep-th/9912268.
- [29] I.G. Halliday and P. Suranyi. Duals of nonabelian gauge theories in d-dimensions. *Phys. Lett.*, B350:189–196, 1995, hep-lat/9412110.
- [30] R. Oeckl and H. Pfeiffer. The dual of pure non-abelian lattice gauge theory as a spin foam model. *Nucl. Phys.*, B598:400–426, 2001, hep-th/0008095.
- [31] J.C. Baez. Spin foam models. *Class. Quant. Grav.*, 15:1827–1858, 1998, gr-qc/9709052.
- [32] C. Itzykson and J. M. Drouffe. *Statistical field theory. Vol. 1: From Brownian motion to renormalization and lattice gauge theory*. Cambridge University Press, Cambridge, 1989.
- [33] F. Conrady. Geometric spin foams, Yang-Mills theory and background-independent models. 2005, gr-qc/0504059.
- [34] P. Menotti and E. Onofri. The action of SU(N) lattice gauge theory in terms of the heat kernel on the group manifold. *Nucl. Phys.*, B190:288, 1981.
- [35] J.W. Cherrington, D. Christensen, and I. Khavkine. Dual computations of non-abelian Yang-Mills on the lattice. 2007, hep-lat/07052629.
- [36] F. Conrady. Dual representation of Polyakov loop in 3d SU(2) lattice Yang-Mills theory. 2007, arXiv:0706.3422 [hep-lat].
- [37] D.J. Gross and W. Taylor. Two-dimensional QCD is a string theory. *Nucl. Phys.*, B400:181–210, 1993, hep-th/9301068.
- [38] D.J. Gross and W. Taylor. Two-dimensional QCD and strings. 1993, hep-th/9311072.